MATHEMATICS (B.Sc - Part I) Paper-II

EXAMPLES ON TANGENT AND NORMALS

J.K. Sinha

Example 1. Find the equation of the tangent and normal at the point (x, y) of the curve $y^2 = 4ax$

Ans: We have $y^2 = 4ax$ ——— (1)

Differentiating this with respect to x on both sides, we have

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

Putting this value in the equation of tangent at (x, y) we have

$$(Y - y) = \frac{2a}{y}(X - x)$$

or $(Y - y) = \frac{2aX}{y} - 2a\frac{x}{y}$

or $Yy - y^2 = 2aX - 2ax$

or $Yy = 2aX + y^2 - 2ax$

$= 2aX + 4ax - 2ax \quad [\because y^2 = 4ax]$

$= 2aX + 2ax = 2a(X + x)$

Hence the equation of the tangent is

$$Yy = 2a(X + x)$$

Now the equation of the normal is

$$(Y - y) = -\frac{1}{\frac{dy}{dx}}(X - x)$$

or $(X - x) + \frac{dy}{dx}(Y - y) = 0$

or $(X - x) + \frac{2a}{y}(Y - y) = 0 \left[\because \frac{dy}{dx} = \frac{2a}{y}\right]$

or $2a(Y - y) + yX - xy = 0$

or $yX + 2aY = 2ay + xy$ is the required equation of the normal.

EXAMPLE II Find the condition that the conics, $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$ shall cut orthogonally.

The equations of the conics are

$$f(x,y)=ax^2+by^2-1=0 \quad \text{——} (1)$$

and

$$\phi(x,y)=a_1x^2+b_1y^2-1=0 \quad \text{——} (2)$$

We know that the condition that they should cut orthogonally at (x,y) is

$$f_x\phi_x+f_y\phi_y=0 \quad \text{——} (3)$$

Now
$$f_x=2ax$$
$$f_y=2by$$
$$\phi_x=2a_1x$$
$$\phi_y=2b_1y$$

$\therefore$ Putting these values in (3) we have

$$aa_1x^2+bb_1y^2=0 \quad \text{——} (4)$$

We shall find the condition after eliminating (x,y) from (1), (2) and (4),

Subtracting (2) from (1) we have

$$(a-a_1)x^2+(b-b_1)y^2=0 \quad \text{——} (5)$$

Comparing (4) and (5) we have

$$\frac{a-a_1}{aa_1}=\frac{b-b_1}{bb_1}$$

or $\frac{1}{a_1}-\frac{1}{a}=\frac{1}{b_1}-\frac{1}{b}$ is the required condition.

EXAMPLE III

If $X\cos a + Y\sin a = p$ touches the curve

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$$ show that

$$(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}}$$

Ans: we have $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1.$

Differentiating w.r.t. x we get $\frac{mx^{m-1}}{a^m}+\frac{my^{m-1}}{b^m}\cdot\frac{dy}{dx}=0$

$$\therefore \frac{dy}{dx}=-\frac{b^m}{a^m}\cdot\frac{x^{m-1}}{y^{m-1}},$$

Hence the equation of the tangent o the given curve at (x,y) is

$$Y-y=\frac{dy}{dx}(X-x) \text{ or, } Y-y=-\frac{b^m}{a^m}\cdot\frac{x^{m-1}}{y^{m-1}}(X-x)$$

$$\text{or } (X-x)\frac{x^{m-1}}{a^m}+(Y-y)\frac{y^{m-1}}{b^m}=0$$

$$\text{or } \frac{Xx^{m-1}}{a^m}+\frac{Yy^{m-1}}{b^m}=\frac{x^m}{a^m}+\frac{y^m}{b^m}=1 \quad \text{——} (1)$$

If $X\cos\alpha + Y\sin\alpha = p$ —— (2) touches the given curve so the equation (1) and (2) are identical' we have thus

$$\frac{\frac{x^{m-1}}{a^m}}{\cos\alpha}=\frac{\frac{y^{m-1}}{b^m}}{\sin\alpha}=\frac{1}{p}$$

$$\text{i.e. } \frac{\frac{x^{m-1}}{a^{m-1}}}{a\cos\alpha}=\frac{\frac{y^{m-1}}{b^{m-1}}}{b\sin\alpha}=\frac{1}{p}$$

$$\therefore \left(\frac{x}{a}\right)^{m-1}=\frac{a\cos\alpha}{p}$$

$$\left(\frac{y}{b}\right)^{m-1}=\frac{b\sin\alpha}{p}$$

$$\therefore \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}}+\left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}=\left(\frac{x}{a}\right)^{\frac{m}{m-1}\cdot m-1}+\left(\frac{y}{b}\right)^{\frac{m}{m-1}\cdot m-1}$$

$$=\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1$$

$$\therefore (a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}=p^{\frac{m}{m-1}}$$

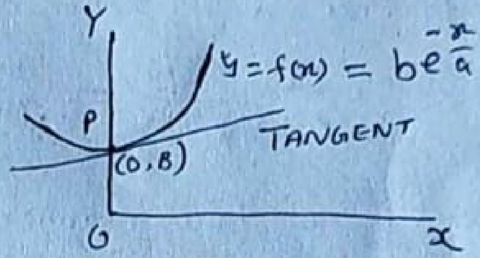
EXAMPLE(4) Find the tangent to the curve $y = be^{-\frac{x}{a}}$ where the curve crosses the axis of y.

This example may also be written as

Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-\frac{x}{a}}$ at the point where the curve crosses the axis of y.

Solution Since the curve crosses the axis of y, so the co-ordinate of the point of intersection are $(0, b)$.

Now differentiating



$y = be^{-\frac{x}{a}}$ w.r.t. x

we have $\frac{dy}{dx} = b\left(-\frac{1}{a}\right)e^{-\frac{x}{a}}$

$= -\frac{b}{a}e^{-\frac{x}{a}} = -\frac{b}{a}e^{-\frac{0}{a}} = -\frac{b}{a}$, at the point $(0, b)$

$\therefore$ The equation of the tangent at $(0, b)$ is

$y - b = -\frac{b}{a}(x - 0)$ or $\frac{y-b}{b} = -\frac{x}{a}$ or $\frac{y}{b} - 1 = \frac{-x}{a}$

i.e. $\frac{x}{a} + \frac{y}{b} = 1$.

Ex(5) Find where the tangent of the curve $ax^2 + 2hxy + by^2 = 1$ is parallel to the axis of x and perpendicular to the axis of x.

Solution $\because ax^2 + 2hxy + by^2 = 1$

Differentiating w.r.t. x, we have

$2ax + 2hy + 2hx \cdot \frac{dy}{dx} + 2by\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

So the tangent will be parallel to x-axis where the line $ax + hy = 0$ intersect the curve similarly the tangent is parallel to y-axis if $hx + by = 0$

$$\left[\because -\frac{dx}{dy} = 0\right]$$